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A criterion for global hyperbolicity of left-invariant Lorentz metrics on Lie groups

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Abstract

We show that every left-invariant Lorentz metric on a non-abelian simply connected Lie group is globally hyperbolic whenever its restriction to the commutator ideal of the Lie algebra is positive definite. We also show that a left-invariant Lorentz metric on the three-dimensional Heisenberg group is globally hyperbolic if and only if its restriction to the center of the Lie algebra is positive definite or degenerate.

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1. Introduction

In many cases, the existence of a globally hyperbolic covering for a compact spacetime M implies the existence of a closed timelike geodesic in M (see [4,5,7,8,13,14,19]). An interesting class of examples of spacetimes admitting globally hyperbolic coverings arises from affine representation of Lie groups, which is in turn closely related to the existence of left-invariant Lorentzian structures (especially left-invariant Lorentzian metrics) on Lie groups. In this context, according to Theorem 3.5 of [2], the three-dimensional Heisenberg group H_3 acts affinely and simply transitively on $\mathbb{R}^3$ in two natural but essentially different ways. The remarkable thing is that one of these actions preserves a flat (and hence, globally hyperbolic) Lorentz metric on $\mathbb{R}^3$ which corresponds to a left-invariant Lorentz metric on H_3 whose restriction to the center of the Lie algebra $\mathcal{H}_3$ is degenerate, and the other one preserves a non-flat Lorentz metric on $\mathbb{R}^3$ which corresponds to a left-invariant Lorentz metric on H_3 whose restriction to the center of $\mathcal{H}_3$ is positive definite. On the other hand, by Theorem 4.2 of [11], if $\langle\,,\,\rangle$ is a left-invariant Lorentz metric on the $(2n+1)$-dimensional Heisenberg group H_{2n+1} whose restriction to the center of the Lie algebra is timelike, then the spacetime $(H_{2n+1}, \langle\,,\,\rangle)$ is not globally hyperbolic. The question then arises of whether $\mathbb{R}^3$ endowed with the non-flat Lorentz metric mentioned above is globally hyperbolic.

In this paper, we will answer this question positively by proving a more general result stating that if G is a non-abelian simply connected Lie group with Lie algebra $\mathcal{G}$, and $\langle\,,\,\rangle$ a left-invariant Lorentz metric on G whose restriction

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to the commutator ideal $\mathcal{G}'$ is positive definite, then the spacetime $(G, \langle\ ,\ \rangle)$ is globally hyperbolic. We then give some applications and derive some consequences of this result. For instance, as a consequence of this result, we find that a non-abelian simply connected nilpotent Lie group endowed with a left-invariant Lorentz metric whose restriction to the center of the Lie algebra is positive definite is globally hyperbolic. We also show by applying the result that the Heisenberg group H_3 endowed with a left-invariant Lorentz metric is globally hyperbolic if and only if the restriction of the metric to the center of the Lie algebra $\mathcal{H}_3$ is positive definite or degenerate.

2. Basic concepts

In this section, we give the necessary background material from tensor and causal geometry and Lie group theory needed in this paper. We refer the reader to [1,15,17,18] for more details than we can include here.

Throughout this paper, a *spacetime* will mean a connected, oriented and time-oriented Lorentz manifold.

2.1. Lorentzian submersions with bounded geometry

Definition 2.1. Let (M, g) and (B, h) be two pseudo-Riemannian manifolds, with $\dim B \leq \dim M$. A mapping $\pi : M \to B$ is a pseudo-Riemannian submersion if it satisfies the following conditions:

(i) π is a smooth submersion.
(ii) The fibers $\pi^{-1}\{p\}$ are pseudo-Riemannian submanifolds of M, for all $p \in B$.
(iii) For any $p \in M$, the mapping $T_p\pi$ is an isometry between the horizontal subspace $T_pM^h = \left(\ker T_p\pi\right)^{\perp}$ of T_pM and the tangent space $T_{\pi(p)}B$.

In the case where (M, g) and (B, h) are both Lorentzian manifolds, we say that π is a *Lorentzian submersion*.

For a pseudo-Riemannian submersion $\pi : (M, g) \to (B, h)$, vectors tangent to fibers are called *vertical*, and those normal to fibers are called *horizontal*. This determines an orthogonal splitting

$$TM = TM^h \oplus TM^v,$$

of the tangent bundle TM of M into horizontal and vertical subbundles TM^h and TM^v, respectively. Each vector field X on B has a unique horizontal lift $\overline{X}$ on M, called the *basic lift* of X. Namely, $\overline{X}$ is horizontal and π-related to X.

There are two (2, 1) tensors which determine the pseudo-Riemannian submersion π locally. The first one is the *integrability tensor* $A : TM^h \times TM^h \to TM^v$ given by

$$A_X Y = \frac{1}{2}[X, Y]^v = (\nabla_X Y)^v,$$

and the other tensor is the *second fundamental form of the fibers* $S : TM^h \times TM^v \to TM^v$ given by

$$S_X V = -(\nabla_V X)^v,$$

where ∇ is the Levi-Civita connection of (M, g).

Note that if $\pi : M \to B$ is a Lorentzian submersion, then the fibers are spacelike, that is, the restriction of g to each fiber is positive definite. It follows that the following definition makes sense.

Definition 2.2. A Lorentzian submersion $\pi : (M, g) \to (B, h)$ is said to have *bounded geometry* if and only if, for any $p \in B$ and vector fields X, Y on B, each of the following conditions is satisfied.

(i) The norm of the operator $S_{\overline{X}_q} : T_qM^v \to T_qM^v$ is bounded, as q ranges over the fiber $\pi^{-1}(p)$.
(ii) $\left|A_{\overline{X}_q}\overline{Y}_q\right|$ is bounded.

Here is a criterion for constructing various globally hyperbolic spacetimes.

Theorem 2.3 (*Walschap; see [20]*). *Let $\pi : (M, g) \to (B, h)$ be a Lorentzian submersion with complete fibers and having bounded geometry. If B is globally hyperbolic with Cauchy hypersurface $\mathcal{S}$, then M is globally hyperbolic with Cauchy hypersurface $\pi^{-1}(\mathcal{S})$.*

2.2. Homogeneous Lorentzian submersions

By Proposition 2.28 in [3], if G is a Lie group acting properly and freely by isometries on a Riemannian manifold (M, g), then there exists a unique Riemannian metric g on $B = M/G$ for which the canonical projection $\pi : M \to B$ is a Riemannian submersion. Inspection of the proof of that proposition shows that this result may be generalized to pseudo-Riemannian submersions with nondegenerate fibers.

Proposition 2.4. *Let (M, g) be a pseudo-Riemannian manifold, and G a Lie group acting properly and freely by isometries on M (hence, the quotient space $B = M/G$ is a manifold and the canonical projection $\pi : M \to B$ is a fibration). If the fibers $\pi^{-1}\{p\} \cong G$, $p \in B$, are nondegenerate with respect to g, then there exists a unique pseudo-Riemannian metric h on B for which π is a pseudo-Riemannian submersion.*

Definition 2.5. If $\pi : M \to B$ is as in the above proposition, then π is said to be *homogeneous.*

Obviously, we have the following useful lemma.

Lemma 2.6. *Any homogeneous pseudo-Riemannian submersion has bounded geometry.*

2.3. Globally hyperbolic spacetimes

Recall that a non-zero tangent vector X to a spacetime (M, g) is said to be spacelike if $g(X, X) > 0$, null (or lightlike) if $g(X, X) = 0$, timelike if $g(X, X) < 0$, and causal (or nonspacelike) if $g(X, X) \leq 0$. For $p \in M$, the causal future $J^+(p)$ of p is the set of points that can be reached from p by a future directed causal curve; the causal past $J^-(p)$ is defined similarly. We say that (M, g) is *strongly causal* if for each $p \in M$, causal curves that start arbitrarily close to p and leave some fixed neighborhood cannot return arbitrarily close to p. We say that (M, g) is *globally hyperbolic* if it is strongly causal and the sets $J^+(p) \cap J^-(q)$ are compact for all $p, q \in M$. A subset $\mathcal{S} \subset M$ is a called a *Cauchy hypersurface* if and only if every inextendible causal curve intersects $\mathcal{S}$ exactly once. A well known result of causal theory asserts that a spacetime M is globally hyperbolic if and only if M admits a Cauchy hypersurface. In this case, M is homeomorphic to $\mathcal{S} \times \mathbb{R}$. A covering $\pi : \widetilde{M} \to M$ is called *regular* if for all $p, q \in \widetilde{M}$ such that $\pi(p) = \pi(q)$, there exists a deck transformation ϕ such that $\phi(p) = q$.

2.4. A remarkable globally hyperbolic left-invariant Lorentz metric on the three-dimensional Heisenberg group

Let G be a connected and simply connected Lie group G with Lie algebra $\mathcal{G}$. Recall that every nondegenerate inner product $\langle\, , \rangle$ on $\mathcal{G}$ induces a left-invariant pseudo-Riemannian metric on G, also denoted by $\langle\, , \rangle$, which is given at each $g \in G$ by

$$\langle X, Y\rangle_g = \left\langle TL_{g^{-1}}X, TL_{g^{-1}}Y\right\rangle, \quad X, Y \in T_gG,$$

where TL_g denotes the derivative of the left translation by g.

Recall also that G is said to be two-step nilpotent if its Lie algebra $\mathcal{G}$ is two-step nilpotent, that is, the commutator (or derived) ideal $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$ is non-trivial and satisfies $[\mathcal{G}, [\mathcal{G}, \mathcal{G}]] = 0$. For background on the Lorentz geometry of two-step nilpotent Lie groups, see [10]. The simplest example of a two-step nilpotent Lie group is the Heisenberg group H_{2n+1} of dimension $2n + 1$. In suitable coordinates, H_{2n+1} is $\mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, t) \cdot \left(z', t'\right) = \left(z + z', t + t' + 2\,\mathrm{Im}\left(zz'\right)\right).$$

Consider the particular case $n = 1$. As usual, the Lie algebra $\mathcal{H}_3$ of H_3 can be identified with the tangent space T_eH_3 to H_3 at the identity e. It has a basis $\{e_1, e_2, e_3\}$ so that $[e_1, e_2] = e_3$, and $[e_1, e_3] = [e_2, e_3] = 0$. In particular, the center of $\mathcal{H}_3$ is $\mathcal{Z} = [\mathcal{H}_3, \mathcal{H}_3] = \mathbb{R}e_3$.

For $g = (x, y, z)$ in H_3, the vectors $e_1, e_2, e_3 \in T_eH_3$ extend to left-invariant vector fields $e_i(g) = TL_g(e_i)$ as follows:

$$e_1(g) = (1, 0, 0), \qquad e_2(g) = (0, 1, x), \qquad e_3(g) = (0, 0, 1).$$

Relative to the basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$ of T_gH_3, we have

$$e_1(g) = \frac{\partial}{\partial x}, \qquad e_2(g) = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad e_3(g) = \frac{\partial}{\partial z}.$$

Now, define the indefinite inner product $\langle\, ,\, \rangle_e$ on $T_e H_3 \cong \mathcal{H}_3$ by setting $\langle e_1, e_1\rangle_e = \langle e_3, e_3\rangle_e = 1$, $\langle e_2, e_2\rangle_e = -1$, and $\langle e_i, e_j\rangle_e = 0$ otherwise (observe that the center $\mathcal{Z} = \mathbb{R}e_3$ is spacelike). This induces a left-invariant Lorentz metric $\langle\, ,\, \rangle$ on H_3 via left translation, which must be geodesically complete (see [6]). In the coordinate system (x, y, z), it is straightforward to check that the left-invariant metric $\langle\, ,\, \rangle$ has the form

$$ds^2 = dx^2 - dy^2 + (dz - x\,dy)^2 .$$

Consider the ideal $\mathcal{K} = \operatorname{span}\{e_1, e_3\}$, and let K be the connected normal Lie subgroup whose Lie algebra is $\mathcal{K}$. The projection

$$\pi : H_3 \to H_3/K \cong \mathbb{R}$$

is a fibration with fibers isomorphic to K. Moreover, at any $g = (x, y, z)$, the tangent space $\mathcal{V}(g)$ to the fiber gK is spanned by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z}$, and the horizontal $\mathcal{H}(g) = \mathcal{V}(g)^{\perp}$ is spanned by $\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$. Thus $\mathcal{H}(g)$ is given by the equations $dx = 0$ and $dz - x\,dy = 0$.

Therefore, by endowing $\mathbb{R} \cong H_3/K$ with the metric $-dy^2$ and recalling that the metric on H_3 is $ds^2 = dx^2 - dy^2 + (dz - x\,dy)^2$, we see that the mapping

$$d_g\pi : \mathcal{H}(g) \to T_{\pi(g)}(H_3/K)$$

is an isometry; or equivalently, the mapping

$$\pi : \left(H_3, ds^2\right) \to \mathbb{R}^1_1 \cong \left(H_3/K, -dy^2\right)$$

is a Lorentzian submersion.

Lemma 2.7. *If γ is a past or future inextendible causal curve in H_3, then so is $\pi \circ \gamma$.*

Proof. Let $\gamma : [0, b) \to H_3$ be a future inextendible causal curve, and write $\gamma(t) = (x(t), y(t), z(t))$. Suppose to the contrary that $\pi \circ \gamma$ is extendible at b. Then, by observing that $\pi \circ \gamma(t) = y(t)$, we may assume that $|\dot{y}(t)|$ is bounded above, say $|\dot{y}(t)| \le 1$. Moreover, since γ is causal, we have $\dot{x}^2 - \dot{y}^2 + (\dot{z} - x\dot{y})^2 \le 0$. It follows that $\dot{x}^2 + (\dot{z} - x\dot{y})^2 \le \dot{y}^2 \le 1$, from which we deduce that $|\dot{x}(t)|$ and $|\dot{z}(t)|$ are bounded above. Hence γ has finite length, and by completeness of the fiber, which follows from the fact that the Lie subgroup K inherits a Riemannian metric, we deduce that $\gamma(t)$ has a limit as $t \to b$, contradicting the hypothesis. The case of a past inextendible causal curve can be dealt with similarly. ■

Theorem 2.8. *The spacetime $(H_3, \langle\, ,\, \rangle)$ is globally hyperbolic.*

Proof. Consider the foliation $\{gK : g \in H_3\}$ of H_3 consisting of left cosets of K. This is a codimension 1, spacelike, left-invariant foliation of H_3. We shall show that each leaf gK is a Cauchy hypersurface or, equivalently, we shall show that for each $c \in \mathbb{R}^1_1$ the level $\pi^{-1}\{c\}$ is a Cauchy hypersurface. Let γ be a past and future inextendible causal curve in H_3. By Lemma 2.7, $\pi \circ \gamma$ is past and future inextendible in $\mathbb{R} \cong H_3/K$. Hence, the image of $\pi \circ \gamma$ coincides with all of $\mathbb{R} \cong H_3/K$. Therefore γ meets each $\pi^{-1}\{c\}$ exactly once, as required. ■

Remark 1. With $(H_3, \langle\, ,\, \rangle)$ as in the above theorem, we constructed in [14] a lattice Γ (i.e., a discrete cocompact subgroup) of H_3 so that the geodesically complete compact spacetime $(H_3/\Gamma, g)$, which admits a globally hyperbolic covering by the theorem above, contains no closed timelike or lightlike geodesics. Here g denotes the Lorentzian metric on H_3/Γ associated with the left-invariant Lorentz metric $\langle\, ,\, \rangle$ on H_3.

3. Main results

In this section, we will give a criterion for a left-invariant pseudo-Riemannian metric on a Lie group to be globally hyperbolic. This extends the result of Theorem 2.8 concerning the special case of H_3 to all non-abelian Lie groups endowed with a left-invariant Lorentz metric whose restriction to the commutator ideal is positive definite. Before proceeding, we state a simple but very useful fact.

Proposition 3.1. *Let G be an abelian, connected, and simply connected Lie group endowed with a left-invariant Lorentz metric $\langle\ ,\ \rangle$. Then, the spacetime $(G, \langle\ ,\ \rangle)$ is globally hyperbolic.*

Proof. It is well known that every left-invariant pseudo-Riemannian metric on an abelian Lie group is geodesically complete and flat. Hence, $(G, \langle\ ,\ \rangle)$ is isometric to Minkowski space $\mathbb{R}^m_1$, where $m = \dim G$. ■

We now give the main result of this paper, which shows that the example of $(H_3, \langle\ ,\ \rangle)$ in the preceding section is an example of a general situation.

Theorem 3.2. *Let G be a non-abelian, connected, and simply connected Lie group with Lie algebra $\mathcal{G}$, and let $\langle\ ,\ \rangle$ be a left-invariant Lorentz metric on G whose restriction to the commutator ideal $\mathcal{G}'$ is positive definite. Then the spacetime $(G, \langle\ ,\ \rangle)$ is globally hyperbolic.*

Proof. By hypothesis, the metric $\langle\ ,\ \rangle$ induces a left-invariant Riemannian metric (i.e., a field of symmetric, positive definite, bilinear forms) on the commutator subgroup G'. Therefore, G' is a homogeneous spacelike submanifold of G. Since G' is normal and acts by isometries (left translations) on G, it follows by Proposition 2.4 of [12] that the quotient group G/G' inherits a unique left-invariant metric g which must be Lorentzian (since G' is spacelike) and such that the canonical projection $\pi : G \to G/G'$ is a Lorentzian submersion. Furthermore, being simply connected and abelian by Lemma 5.5 of [12], the spacetime $\left(G/G', g\right)$ is isometric to Minkowski space $\mathbb{R}^m_1$, where $m = \dim G - \dim G'$. On the other hand, since G' is a homogeneous space, the fibers of π (which are the cosets gG', $g \in G$) are geodesically complete. Since π is homogeneous (see Lemma 2.6), the submersion π has bounded geometry. It follows that all the conditions of Theorem 2.3 are fulfilled for π. Hence $(G, \langle\ ,\ \rangle)$ is globally hyperbolic, and $\pi^{-1}(\mathcal{S})$ is a Cauchy hypersurface in G whenever $\mathcal{S}$ is a Cauchy hypersurface in $G/G' \cong \mathbb{R}^m_1$. This completes the proof of the theorem. ■

4. Applications

As an immediate consequence of Theorem 3.2, we have the following result.

Theorem 4.1. *Let N be a non-abelian, connected, and simply connected nilpotent Lie group with Lie algebra $\mathcal{N}$, and let $\langle\ ,\ \rangle$ be a left-invariant Lorentz metric on N whose restriction to the center $\mathcal{Z}$ of $\mathcal{N}$ is positive definite. Then the spacetime $(N, \langle\ ,\ \rangle)$ is globally hyperbolic.*

Proof. Since N is non-abelian and nilpotent, the commutator subgroup N' is contained in the center Z of N. Thus, the restriction of $\langle\ ,\ \rangle$ to the commutator ideal $\mathcal{N}'$ is positive definite. The result then follows straightforwardly from Theorem 3.2. ■

Remark 2. Consider a two-step nilpotent Lie group N with a left-invariant Lorentz metric $\langle\ ,\ \rangle$ whose restriction to the center of the Lie algebra of N is positive definite. Then, according to the above theorem, the spacetime $(N, \langle\ ,\ \rangle)$ is globally hyperbolic. On the other hand, given any lattice Γ in N, we have shown in [10] that every closed timelike or lightlike geodesic in the geodesically complete compact Lorentz two-step nilmanifold N/Γ lifts to a geodesic in N that is everywhere orthogonal to the center of N. Thus, by suitably choosing Γ, we can generalize the example of H_3/Γ that we mentioned in Remark 1.

For when the restriction of the metric to the center is indefinite, we have shown in [11] (by using explicit computations in terms of Ricci curvatures) the following theorem.

Theorem 4.2. *If $\langle\ ,\ \rangle$ is a left-invariant Lorentz metric on the Heisenberg group H_{2n+1} for which the center is timelike, then the spacetime $(H_{2n+1}, \langle\ ,\ \rangle)$ is not globally hyperbolic.*

In the case where the restriction of the metric $\langle\ ,\ \rangle$ to the center of H_3 is degenerate, we will show that the spacetime $(H_3, \langle\ ,\ \rangle)$ is globally hyperbolic. This will be stated as a corollary of the following result that is of interest in itself.

Theorem 4.3. *A left-invariant Lorentz metric $\langle\ ,\ \rangle$ on the Heisenberg group H_3 is flat if and only if its restriction to the center $\mathcal{Z}$ of $\mathcal{H}_3$ is degenerate.*

Proof. We have proved in Theorem 1 of [9] that a connected two-step nilpotent Lie group N admits a flat left-invariant Lorentz metric if and only if N is a trivial central extension of H_3; and that the restriction of any such metric to the factor H_3 is Lorentzian with degenerate center. In particular, if $\langle\, ,\, \rangle$ is a flat left-invariant Lorentz metric H_3, then the restriction of $\langle\, ,\, \rangle$ to the center $\mathcal{Z}$ of $\mathcal{H}_3$ is degenerate.

Conversely, suppose that $\langle\, ,\, \rangle$ is a left-invariant Lorentz metric H_3 whose restriction to the center $\mathcal{Z}$ of $\mathcal{H}_3$ is degenerate. Since $\dim \mathcal{Z} = 1$, there exists a null vector e_0 which spans $\mathcal{Z}$ (i.e., $\langle e_0, e_0\rangle = 0$, and $\mathcal{Z} = \mathbb{R}e_0$). Since $\langle\, ,\, \rangle$ is nondegenerate, we can find a vector e_1 in $\mathcal{H}_3 \setminus \mathcal{Z}$ such that $\langle e_1, e_1\rangle = 0$ and $\langle e_0, e_1\rangle = 1$.

Let W be the subspace of $\mathcal{H}_3$ spanned by e_0 and e_1. By construction, W is a timelike plane section in $\mathcal{H}_3$. Therefore, let e_2 be a spacelike unit vector in $\mathcal{H}_3$ which spans the orthogonal complement $W^\perp$ of W in $\mathcal{H}_3$ (i.e., $\langle e_2, e_2\rangle = 1$, and $W^\perp = \mathbb{R}e_2$). Summarizing, we find that $\{e_0, e_1, e_2\}$ is a pseudo-orthonormal basis for $\mathcal{H}_3$. By setting $[e_1, e_2] = \alpha e_0$, with $\alpha \neq 0$, and recalling (see [16], page 310) that the Levi-Civita connection ∇ of a left-invariant metric is given by the formula

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left\{\langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle\right\},$$

we easily check that the only non-zero covariant derivatives of the metric $\langle\, ,\, \rangle$ are

$$\nabla_{e_1} e_1 = -\alpha e_2, \quad \text{and} \quad \nabla_{e_1} e_2 = \alpha e_0. \tag{4.1}$$

Recalling that the Riemann curvature tensor $\mathcal{R}$ is defined by

$$\mathcal{R}(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y],$$

and using (4.1), it follows that

$$\begin{aligned} \mathcal{R}(e_0, e_i) &= -\nabla_{e_0}\nabla_{e_i} + \nabla_{e_i}\nabla_{e_0} \\ &= 0, \quad \text{for } i = 1, 2 \\ \mathcal{R}(e_1, e_2) &= \nabla_{[e_1,e_2]} - \nabla_{e_1}\nabla_{e_2} - \nabla_{e_2}\nabla_{e_1} \\ &= \nabla_{e_0} - \nabla_{e_1}\nabla_{e_2} - \nabla_{e_2}\nabla_{e_1} \\ &= 0, \end{aligned}$$

which means that the metric $\langle\, ,\, \rangle$ is flat. ■

Remark 3. We point out that the preceding result cannot be extended to left-invariant Lorentz metrics on the Heisenberg group H_{2n+1}, for $n > 1$. This is a consequence of the fact that H_{2n+1} admits a flat left-invariant Lorentz metric if and only if $n = 1$ (see Corollary 1 of [9]).

Corollary 4.4. *Let $\langle\, ,\, \rangle$ be a left-invariant Lorentz metric on the Heisenberg group H_3. If the restriction of $\langle\, ,\, \rangle$ to the center $\mathcal{Z}$ of $\mathcal{H}_3$ is degenerate, then the spacetime $(H_3, \langle\, ,\, \rangle)$ is globally hyperbolic.*

Proof. According to the preceding theorem, the spacetime $(H_3, \langle\, ,\, \rangle)$ is flat. Since every left-invariant pseudo-Riemannian metric on a two-step nilpotent Lie group is geodesically complete (see [6]), it follows that $(H_3, \langle\, ,\, \rangle)$ is isometric to Minkowski space $\mathbb{R}^3_1$; and hence globally hyperbolic. ■

Combining Theorems 3.2 and 4.2, and Corollary 4.4 yields the following result.

Theorem 4.5. *Let $\langle\, ,\, \rangle$ be a left-invariant Lorentz metric on the Heisenberg group H_3. Then, the spacetime $(H_3, \langle\, ,\, \rangle)$ is globally hyperbolic if and only if the restriction of the metric $\langle\, ,\, \rangle$ to the center $\mathcal{Z}$ of $\mathcal{H}_3$ is positive definite or degenerate.*

Proof. Suppose the spacetime $(H_3, \langle\, ,\, \rangle)$ is globally hyperbolic. Then, by Theorem 4.2, the restriction of the metric $\langle\, ,\, \rangle$ to the center $\mathcal{Z}$ of $\mathcal{H}_3$ is positive definite or degenerate.

Conversely, if the restriction of metric $\langle\, ,\, \rangle$ to $\mathcal{Z}$ is positive definite or degenerate, then the spacetime $(H_3, \langle\, ,\, \rangle)$ is globally hyperbolic according to Theorem 3.2 and Corollary 4.4, respectively. ■

Remark 4. Let $n > 1$, and let $\langle\, ,\, \rangle$ be a left-invariant Lorentz metric on the Heisenberg group H_{2n+1} whose restriction to the center of $\mathcal{H}_{2n+1}$ is degenerate. The question then arises of whether the spacetime $(H_{2n+1}, \langle\, ,\, \rangle)$ is globally hyperbolic. We conjecture that this is true.

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